Jonas Kremer WS 2017/2018

Probability theory

Exercise Sheet 9

Exercise 1 *(4 Points)*
Let (E, d) be a metric space and let $V : E \to \mathbb{R}_+$ be continuous. For $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{P}(E)$ and $\mu \in \mathcal{P}(E)$ suppose that $\mu_n \to \mu$ weakly as $n \to \infty$ and

$$\sup_{n\geqslant 1} \int_E V(x)\mu_n(\mathrm{d}x) < \infty.$$

Prove that

$$\int_E V(x)\mu(\mathrm{d}x) \leqslant \sup_{n\geqslant 1} \int_E V(x)\mu_n(\mathrm{d}x) < \infty.$$

Hint: Use the approximation $V_R(x) := \max\{V(x), R\}$ where $R > 0$. Then V_R is continuous and bounded.

Exercise 2 *(6 Points)*
For given $\sigma > 0$ and $b \in \mathbb{R}$ let

$$\mu_{\sigma,b}(\mathrm{d}x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{-(x-b)^2}{2\sigma^2}\right\} \mathrm{d}x.$$

(a) For a polynomial p define

$$F_p(\sigma, b) := \int_{\mathbb{R}} p(x)\mu_{\sigma,b}(\mathrm{d}x).$$

Prove by 'direct computation' and exercise sheet 1 that F_p is continuous in σ and b.

(b) Deduce from the continuity of F_p that

$$\int_{\mathbb{R}} f(x)\mu_{\sigma,b}(\mathrm{d}x) \to \int_{\mathbb{R}} f(x)\mu_{\sigma',b'}(\mathrm{d}x), \quad (\sigma, b) \longrightarrow (\sigma', b') \tag{1}$$

for all continuous $f : \mathbb{R}^d \to \mathbb{R}$ such that there exists a constant $c > 0$ and $\kappa \geqslant 1$ with $|f(x)| \leqslant c(1 + |x|^\kappa)$ for all $x \in \mathbb{R}^d$.

(c) Prove that for any $0 < \sigma_* < \sigma^*$ and $b_* < b^*$

$$\lim_{R\to\infty} \sup_{(\sigma,b)\in[\sigma_*,\sigma^*]\times[b_*,b^*]} H_R(\sigma,b) = 0$$

where the function H_R is for $R > 0$ given by

$$H_R(\sigma,b) = \int_{\mathbb{R}} \mathbb{1}_{|x|\geq R}(x)|x|^\kappa \mu_{\sigma,b}(\mathrm{d}x).$$

Deduce (1) from this.

Exercise 3 *(4 Points)*
For $a > 0$ let

$$\mu(dx) = \frac{1}{a}\left(1 - \frac{|x|}{a}\right)\mathbb{1}_{[-a,a]}(x)dx.$$

Show that the characteristic function $\widehat{\mu}$ is for $\xi \in \mathbb{R}$ given by

$$\widehat{\mu}(\xi) = \frac{2(1-\cos(a\xi))}{a^2 - \xi^2}.$$

Exercise 4 *(4 Points)*
Use characteristic functions to prove that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Hint: Let $\varepsilon \geqslant 0$ and X_ε be a $\mathbb{Z}$-valued random variable such that

$$\mathbb{P}\left(X_\varepsilon = k\right) = \begin{cases} 0, & \text{if } k = 0, \\ c(\varepsilon)\frac{e^{-\varepsilon|k|}}{k^2}, & \text{if } k \neq 0, \end{cases}$$

where $c(\varepsilon) > 0$ is fixed and given by $\sum_{k=-\infty}^{\infty} \mathbb{P}(X_\varepsilon = k) = 1$. Compute the second derivative of

$$\varphi_{X_\varepsilon}(\xi) = \mathbb{E}(e^{i\xi X_\varepsilon}), \quad \xi \in \mathbb{R}$$

and evaluate φ_{X_ε} in $\varepsilon = 0$ in order to show that $c(0) = 3/\pi^2$.